II.) Fluctuation Theory

→ The topic of Fluctuation Theory, our second topic, is concerned with:

- generalized ideas of correlation and response, particularly linear response ⇒ { generalized susceptibility / memory function }
- the relation between fluctuations, noise and dissipation ⇒ fluctuation-dissipation theorem
- basic ideas of random walks and diffusion. ⇒ Langevin equation

→ Fluctuation Theory can be formulated in a very generalized, model independent way (a/a flux-force relations) but better to first revisit simple paradigm of Brownian Motion

⇒

a.) Brownian Motion: A Case Study in Fluctuation Theory and Random Walks.

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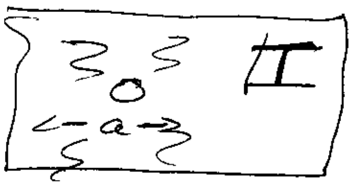
Ref: Chandrasekhar Review, RMP 1943

→ What's the Physics



- small particle immersed in thermally conducting fluid at T, which exhibits thermal fluctuations

- thermal fluctuations "kick" particle ⇒ random, noise force acts on particle : $\tilde{f}$

- scales: $(\Delta x)_{Th} \ll a \ll L$

  $(\Delta x)_{Th}$: typical scale of fluctuation mode

  a: particle size

  L: system size

⇒ noise is spatially homogeneous.

- dissipative system ⇒ drag of fluid on particle is important.

so

$$M \frac{d\underline{v}}{dt} = f_{\text{friction}} + f_{\substack{\text{thermal}\\ \text{noise}}}$$

$$= \underbrace{-\gamma_s}_{\text{Stokes drag}} \underbrace{\left(v - \underline{V}_{\text{fluid}}(\underline{x}, t)\right)}_{\text{frictional slippage force.}} + f_{\substack{\text{thermal}\\ \text{noise}}}$$

for $M\omega \ll \gamma_s \iff$ dissipation dominated limit (equivalent to inertialess particle)

$$\Rightarrow \boxed{\frac{d\underline{v}}{dt} = -\gamma \underline{v} + \underline{\tilde{a}}_{thermal}}$$

Langevin Equation

$\gamma = \gamma_s / M$

$\gamma_s = 6\pi \eta a \rightarrow$ Stokes Drag.

η: fluid viscosity (mass dependent)

a: particle size

$\therefore$ minimal problem:

$$\frac{d\underline{v}}{dt} = -\gamma \underline{v} + \underline{\tilde{a}}_{thermal}$$

$-\gamma \underline{v}$: dissipation

$\underline{\tilde{a}}_{thermal}$: noise $\rightarrow$ "Random"

[additive noise]

typical questions:

- $\rightarrow$ what is $\langle \underline{v}(0)\,\underline{v}(\tau)\rangle$? [correlation function]
- $\rightarrow$ determine $\langle \tilde{v}^2\rangle$, $\langle \tilde{v}^2\rangle_\omega$ — spectrum [level]
- relate $\langle \tilde{v}^2\rangle_\omega$, $\langle \tilde{a}^2\rangle_\omega$, γ (Fluctuation – Dissipation)

→ Basic Time Scales

- $\tau_{ac} = \tau_{autocorrelation}$ → self-correlation time of random force
- γ^{-1} → velocity relaxation time (i.e. damping time)
- T_{macro} → any macroscopic time scale i.e. time to drift accross box.

and order: $\boxed{\tau_{ac} \ll \gamma^{-1} \ll \tau_{macro}}$ → key ordering

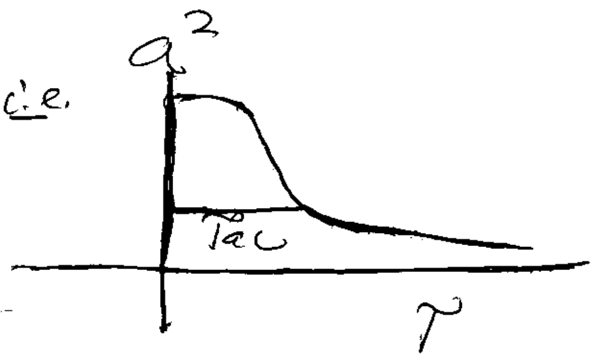
→ Meaning of "Random" and "τ_{ac}"

- τ_{ac} is self-correlation time of $\tilde{a}$, i.e.

$$\langle \tilde{\underline{a}}(t)\, \tilde{\underline{a}}(t+\tau)\rangle = \text{(auto) correlation function of } \tilde{a}$$
$$= \langle \tilde{a}^2(\tau)\rangle \text{ for stationary process}$$

(no bias forward/back in temporal evolution)

i.e.



τ_{ac} is decay time of self-correlation function

∴ → τ_{ac} measures duration time of a "random kick"

(*)

$\tau_{ac} < \gamma^{-1} < T_{macro}$ ⇒ many kicks in motion damping time

→ τ_{ac} ↔ bandwidth of forcing spectrum...

i.e. Wiener-Khinchine Theorem:

$$\boxed{\langle \tilde{f}(t)\,\tilde{f}(t+\tau)\rangle = \int e^{-i\omega\tau}\,|f(\omega)|^2\,d\omega/2\pi}$$

auto-correlation function — spectral density

true for stationary process

Proof: stationarity ⇒ $\langle \tilde{f}_\omega \tilde{f}_{\omega'}\rangle = |f(\omega)|^2\,\delta(\omega+\omega') \equiv \langle \tilde{f}^2\rangle_\omega\,\delta(\omega+\omega')$

i.e. $\langle \tilde{f}^2\rangle_\omega$ is F.T. (in τ) of correlation function

of course: $\int d\omega\,\langle \tilde{f}^2\rangle_\omega = \int d\omega\,|f(\omega)|^2$

strength parameter

usually: $$\langle \tilde{f}^2 \rangle_\omega = \frac{\tilde{f}_0^2 \, \Delta\omega}{(\omega - \omega_0)^2 + \Delta\omega^2}$$

centroid frequency

bandwidth linewidth

$\therefore$

a.) $\langle \tilde{f}(t) \, \tilde{f}(t+\tau) \rangle \sim f_0^2 \, e^{-i\omega_0 \tau} \, e^{-|\Delta\omega \, \tau|}$

oscillation

decay of correlation

so $\boxed{\tau_{ac} = 1/\Delta\omega}$

bandwidth

b.) $\Delta\omega \to 0$, $|f(\omega)|^2 \to \delta(\omega - \omega_0)$

coherency $\to$ narrow bandwidth case

c.) $\Delta\omega \to \infty \Rightarrow |f(\omega)|^2 = f_0^2/\Delta\omega$

$$\langle \tilde{f}(t) \, \tilde{f}(t+\tau) \rangle = \frac{f_0^2}{\Delta\omega} \delta(\tau)$$

"delta correlated" limit.

→ re: What does "Random" Mean?

Consider what response to noise appears ...

$$\frac{dv}{dt} + \gamma v = \tilde{a}(t)$$

$$(-i\omega + \gamma) v_\omega = a_\omega$$

$$\therefore \quad |\tilde{v}(\omega)|^2 = |\tilde{a}(\omega)|^2 \Big/ \left[\omega^2 + \gamma^2\right]$$

$$= \underbrace{|R(\omega)|^2}_{\text{response function.}} \; |\tilde{a}(\omega)|^2$$

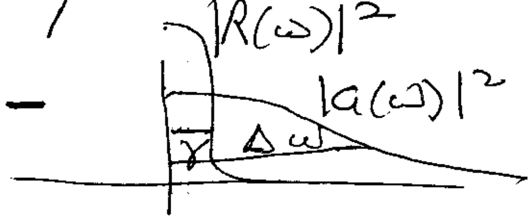
↳ forcing spectrum (i.e. Lorentzian, as before)

Effectively, "Random" ⟺

- $\tilde{a}$ distributed statistically (i.e. quasi-Gaussian, etc.)

- $\Delta\omega \gg \gamma$ i.e. bandwidth exceeds response rate.

why this convention?



$\Rightarrow$ can take $|a(\omega)|^2$ $\sim$ flat

$\therefore$

- can take noise spectrum as effectively "white" $\Leftrightarrow$ constant in ω.

N. B. Of course, can speak of "colored noise"
"Colored noise" $\rightarrow$ statistically distributed $\tilde{a}$
$\rightarrow$ non-flat spectrum
i.e. $\langle \tilde{a}^2 \rangle_\omega \sim \omega^{-\rho}$

"Colored noise" $\Leftrightarrow$ $\begin{cases} \text{effective memory} \\ \text{range of time scales} \end{cases}$

$\rightarrow$ can become tricky .....

- "white noise" is classic example $\Rightarrow$ only time scale is system response.

Which brings us to....

→ the answer: velocity correlation function

In $\Delta\omega \to \infty$, (white noise) limit:

$$\langle v(t)\, v(t+\tau)\rangle = \int \frac{e^{-i\omega\tau}}{2\pi} \frac{|a_0|^2 \, d\omega}{\Delta\omega\,(\omega^2+\gamma^2)}$$

$$\sim \frac{|a_0|^2}{\Delta\omega} \frac{e^{-\gamma|\tau|}}{\gamma}$$

and

$$\langle v(t)\, v(t)\rangle = \langle v(0)^2\rangle = |a_0|^2/\Delta\omega\gamma$$

→ but particle is in thermal fluid, at thermal equilibrium, at temperature T,

$$\frac{M\langle v^2\rangle}{2} = \frac{k_B T}{2}$$

but

$$\frac{M\langle v^2\rangle}{2} = \frac{M}{2}\frac{|a_0|^2}{\Delta\omega\gamma}$$

so....

$$\boxed{a_0^2/\Delta\omega\gamma = k_B T}$$

$$\Rightarrow \boxed{\underbrace{\frac{a_0^2}{\Delta\omega}}_{\text{noise intensity}} = \underbrace{\gamma}_{\text{damping}} \underbrace{k_B T}_{\text{temperature}}}$$

$\Rightarrow$ primitive form of "Fluctuation - Dissipation Theorem".

i.e. more rigorous form : (will prove)

$$\underbrace{x(\omega)}_{\text{displacement}} = \underbrace{\alpha(\omega)}_{\text{response function}} \underbrace{f(\omega)}_{\text{forcing}}$$

i.e. susceptibility $\leftrightarrow \epsilon(k,\omega)$, etc.

$$\alpha = \alpha_{real} + i\,\alpha_{IM}.$$

then F.-D. T. relation is :

fluctuating intensity

$$\langle x^2\rangle_\omega = \frac{2T}{\omega}\,\underbrace{\mathrm{Im}\,\alpha(\omega)}_{\text{damping}}$$

→ temp.

$\hbar\omega \ll k_B T$

or in QM case:

$$\langle x^2\rangle_\omega = \hbar\,\mathrm{Im}\,\alpha(\omega)\coth\frac{\hbar\omega}{2k_BT}$$

operator expectation

$\hbar\omega/k_BT$ finite

→ basically same content as simple case....
Will demonstrate general case next week via consideration of generalized susceptibility.

→ Essence of F-D. Thm:

- temperature, dissipation rate ($\mathrm{Im}\,\alpha$) and fluctuation intensity "locked" together at thermal equilibrium, via.

$$(\text{Intensity}) \sim (\text{Damping Rate})\,T$$

- pick 2 + equilibrium ⇒ 3rd.

- at equal T, more damping ⇒ stronger intensity fluctuations.

→ Related: Random Processes and Diffusion – A Review

→ Fundamental: Central Limit Theorem

Meaning: If:

a) observation "error" x is accumulation of large number of small errors ($N \to \infty$)

b) moments of small error pdf exist

then Total Error X follows Gaussian or "Normal" Distribution.

Precise Statement:

→ Consider a sum of $N \gg 1$ independent random variables (increments)

$\Delta \underline{x}_1, \Delta \underline{x}_2, \ldots \Delta \underline{x}_N$.

take: $$x_n = \sum_{i=1}^{N} \Delta x_i$$ (accumulated increment)

$$s_n^2 = \sum_{i=1}^{N} \sigma_i^2$$

where: $\langle \Delta x_i \rangle = 0$

$\langle \Delta x_i^2 \rangle = \sigma_i^2$ → key proviso: 2nd moment exists.

then given 'appropriate conditions' on Δx_i,

⇒ Pdf of $y_n = x_n / S_n$ → Gaussian Distribution as $N \to \infty$

i.e. $\lim_{N \to \infty} \text{Pdf}(x_n/S_n) \to 1/\sqrt{2\pi}\, \exp(-y^2/2)$

"Appropriate Conditions":

- second moment of each random step/variable pdf exists

→ THIS IS CRUCIAL ....

- N random variables, Δx_i "all alike", i.e. no special few 'bias' or 'dominate' the others ....

Example: IMPORTANT

- net displacement of Brownian particle is sum of many, successive small displacements

i.e. $\tau_{co} < \gamma^{-1} < T_{macro}$

$\therefore$

- expect distribution of <u>net</u> <u>displacement</u>

$$\Delta x_N = \sum_{i=1}^{N} \Delta x_i$$

to be Gaussian (as $N \to \infty$) even if a random walk description is <u>not</u> valid for short intervals of time.

N.B.: Explains why macro / long time dynamics of many systems are Gaussian, even if micro dynamics are <u>not</u> Gaussian.

why Gaussian random processes are ubiquitous ?!

→ Machinery for Gaussian Processes.

- If random variable x has pdf $f(x)$ s/t

$$\langle x^n \rangle = \int_{-\infty}^{+\infty} dx \; x^n f(x)$$

then can define characteristic function

$$\Phi(\varepsilon) = \langle e^{i\varepsilon x} \rangle = \int_{-\infty}^{+\infty} dx \; f(x) \, e^{i\varepsilon x}$$

$\Phi(\varepsilon) \Leftrightarrow$ Fourier transform of pdf; obviously of great interest!

- can relate to moments:

$$\langle x^n \rangle = i^{-n} \left[\left(\frac{d}{d\varepsilon} \right)^n \Phi(\varepsilon) \right]_{\varepsilon \to 0}$$

so equivalent to:

$$\Phi(\varepsilon) = \sum_{n=0}^{\infty} \frac{(i\varepsilon)^n}{n!} \langle x^n \rangle$$

i.e. sum of moments defines $\Phi(\xi)$ and thus pdf.

$\leftrightarrow$ knowledge of all moments equivalent to knowledge of pdf, via characteristic function.

For Gaussian/Normal distribution:

$$f(x) = 1/\sqrt{2\pi\sigma^2}\, \exp\left(-(x-m)^2/2\sigma^2\right)$$

$\Rightarrow$

$$\Phi(\xi) = \exp\left[im\xi - \sigma^2\xi^2/2\right]$$

Can define cumulant:

$$\Phi(\xi) = e^{\psi(\xi)}$$

$\psi(\xi)$ → cumulant function

$$\psi(\xi) = \ln \Phi(\xi)$$

Now:

- $$\psi(\xi) = \sum_{n=1}^{\infty} \frac{(i\xi)^n}{n!} \langle x^n \rangle_c$$

n^{th} cumulant

- i.e. $\langle x \rangle_c = \langle x \rangle$

$$\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x^3 \rangle_c = \langle x^3 \rangle - 3\langle x_c \rangle \langle x^2 \rangle - \langle x \rangle_c^3$$

etc.

- note analogy:

Eqbrm Stat Mech

characteristic function $\Phi(\xi) = \langle e^{i\xi x} \rangle \leftrightarrow$ Partition Function Z

Cumulant function $\psi(\xi) = \ln \Phi \leftrightarrow$ Free Energy F

→ Fluctuation Theory II

a.) Revisiting the Fluctuation–Dissipation Theorem
(Ref: Landau, Lifshitz, Stat. Physics)

To generalize the Fluctuation–Dissipation Theorem, need:
→ generalized susceptibility
→ Kramers–Kronig Relations
→ F.D. Thm.

1.) Generalized Susceptibility (extends to QM systems)

Each physical quantity associated with a thermal fluctuation in a system can be associated with a perturbation in the Hamiltonian, i.e.

$$H = H_0 + \hat{V}$$

where $\hat{V} = -\underbrace{\hat{x}}_{\substack{\text{operator}\\\text{associated}\\\text{with physical}\\\text{quantity}}} f(t)$ → generalized force (fctn. of time)

Now, $\overline{x} \equiv$ mean, in quantum sense, of state of system → i.e. expected response

Then,

$$\underbrace{\bar{X}(t)}_{\substack{\text{system} \\ \text{response to} \\ \text{perturbation}}} = \hat{\alpha} f = \int_0^\infty \underbrace{\alpha(\tau)}_{\substack{\text{generalized} \\ \text{susceptibility}}} \underbrace{f(t-\tau)}_{\text{forcing}} \, d\tau$$

(linear)

→ determined by system

In ω space;

$$\underbrace{\bar{X}(\omega)}_{\text{response}} = \underbrace{\alpha(\omega)}_{\text{susceptibility}} \underbrace{f(\omega)}_{\text{forcing}}$$

i.e.

$$\underbrace{\underline{D}(\omega)}_{\substack{\text{displacement} \\ \text{field}}} = \underbrace{\epsilon(\omega)}_{\substack{\text{dielectric} \\ \text{function}}} \underbrace{\underline{E}(\omega)}_{\substack{\text{electric} \\ \text{field}}}$$

or

$$\underbrace{X(\omega)}_{\substack{\text{oscillator} \\ \text{displacement}}} = \underbrace{\alpha(\omega)}_{\substack{\text{linear} \\ \text{response}}} \underbrace{f(\omega)}_{\text{forcing}}$$

$$\begin{cases} m\dfrac{d^2x}{dt^2} + 2m\gamma\dfrac{dx}{dt} + kx = f(t) \\ \alpha(\omega) = \dfrac{f/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega} \end{cases}$$

General Properties of Susceptibility:

- $\alpha(\omega)$ complex $\quad \alpha(\omega) = \alpha_r(\omega) + i\alpha_{IM}(\omega)$
$= \alpha'(\omega) + i\alpha''(\omega)$

- $\alpha(-\omega) = \alpha(\omega)^*$

from $\alpha(\omega) = \int_0^\infty \alpha(t)\, e^{i\omega t}\, dt$

so $\begin{cases} \alpha_r(-\omega) = \alpha_r(\omega) \\ \alpha_{IM}(-\omega) = -\alpha_{IM}(-\omega) \end{cases}$

- $\alpha(-\omega^*) = \alpha^*(\omega)$

- Meaning of $\alpha_{IM} \leftrightarrow$ Dissipation!

Now, $\underbrace{dE/dt}_{\text{energy evolution}} = \overline{\partial H/\partial t} \rightarrow$ rate of energy absorption

so mean rate of work done by external force is:

$$\frac{dE}{dt} = +\overline{\frac{\partial H}{\partial t}} = -\bar{x}\,\frac{df(t)}{dt}$$

$\rightarrow$ rate of change of external force.

Now, $\frac{dE}{dt} = \left\langle -\frac{1}{2}\left[\alpha(\omega) f_0 e^{-i\omega t} + \alpha(-\omega) f_0^* e^{i\omega t}\right]\frac{1}{2}\left[-i\omega f_0 e^{-i\omega t} + i\omega f_0^* e^{i\omega t}\right]\right\rangle_t$

(single ω)

dissipation rate

$$\therefore \frac{dE}{dt} = \underset{\downarrow}{Q} = -\frac{|f_0|^2}{4}\left\{\alpha(-\omega)(-i\omega) + i\omega\,\alpha(\omega)\right\}$$

$$= \frac{|f_0|^2}{4}\, 2\,\omega\, \alpha_{IM}(\omega)$$

$$\boxed{Q = \frac{|f_0|^2}{2}\,\omega\,\alpha_{IM}(\omega)}$$

→ imaginary part $\alpha \leftrightarrow$ dissipation

→ as dissipation is absolutely necessary can conclude

$\mathrm{Im}\,\alpha(\omega) > 0$, for <u>all</u> ω.

[? Can one formulate F-D. Thm for unstable system?
(i.e. system with growing modes)]

- $\alpha(\omega)$ is single valued, regular function <u>everywhere</u> in UHP.

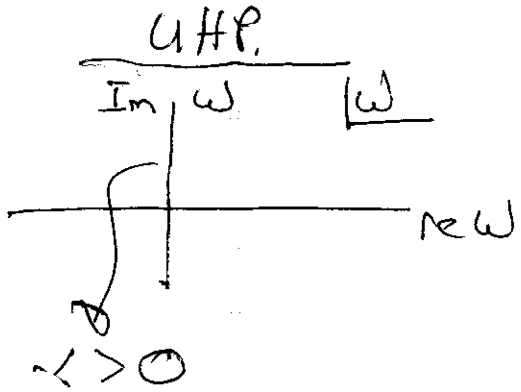
i.e. $\alpha(\omega) = \int_0^\infty e^{i\omega t}\alpha(t)\,dt$

finiteness, <u>causality</u>

Now, can prove the following theorem:

[See L+L Stat. Phys. pgs 380-381 for proof]

Thm: The function $\alpha(\omega)$ does not take real values at any finite point in the UHP except on the imaginary axis, where it decreases monotonically from $\alpha_0 > 0$ at $\omega = i0$ to $\alpha = 0$ at $\omega \to i\infty$. Thus, $\alpha(\omega)$ has no zeroes in



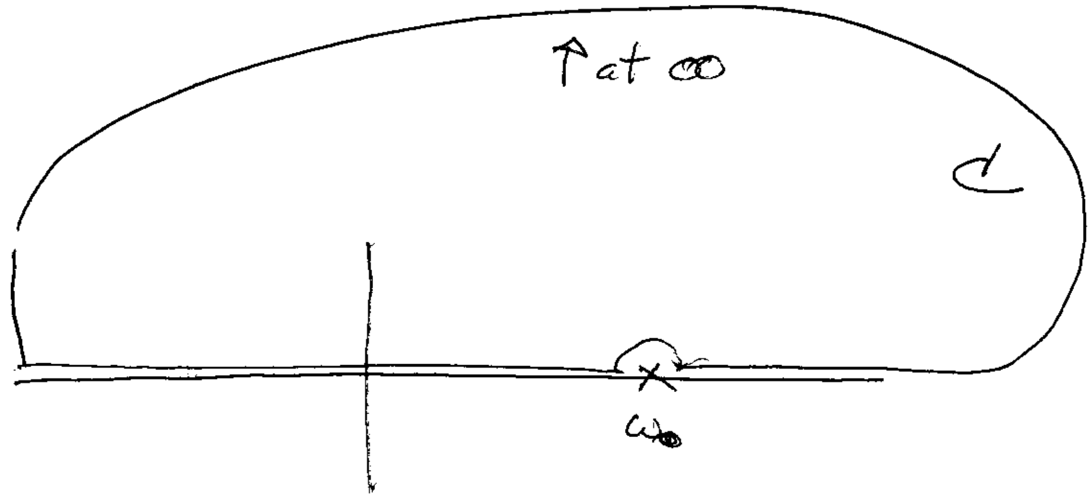
Point: $\alpha(\omega)$ has no zeroes for $\mathrm{Im}\,\omega > 0$.
i.e. zero is "real" value!

As a consequence of the theorem, can prove the Kramers-Kronig Relations.

b.) Kramers-Kronig Relations

Define: $$I = \int_C \frac{\alpha(\omega)}{\omega - \omega_0}\, d\omega$$

where contour C specified by:



Now $I = \int d\omega \, \frac{\alpha(\omega)}{\omega-\omega_0}$

and

→ $\omega \to \infty \Rightarrow \alpha \to 0$

→ $\alpha/\omega-\omega_0 \underset{\omega\to\infty}{\to} 0$ more rapidly than $1/\omega$

$\therefore$ I converges

→ Since the point ω_0 is excluded and Thm $\Rightarrow$ $\alpha(\omega)$ is regular in UHP

$\Rightarrow$ $I = 0$!

but $1/\omega-\omega_0 = \frac{P}{\omega-\omega_0} - i\pi\,\delta(\omega-\omega_0)$

so $I = \int_{-\infty}^{+\infty} d\omega \, \frac{\alpha(\omega)}{\omega-\omega_0} = P\int_{-\infty}^{+\infty} d\omega \, \frac{\alpha(\omega)}{\omega-\omega_0} - i\pi\,\alpha(\omega_0)$

as semicircle-at-infinity contribution vanishes.

So

Have $i\pi\alpha(\omega_0) = P\int_{-\infty}^{+\infty} \frac{\alpha(\omega)}{\omega-\omega_0}\, d\omega$

Now, simply equate real and imaginary parts of above, noting: $\alpha(\omega) = \alpha_r(\omega) + i\,\alpha_{IM}(\omega)$

$$\therefore \boxed{\begin{aligned} \alpha_{real}(\omega) &= \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\alpha_{IM}(y)}{y-\omega}\, dy \\ \alpha_{IM}(\omega) &= -\frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\alpha_{real}(y)}{y-\omega}\, dy \end{aligned}}$$

Note:

→ these constitute Kramers - Kronig Relations ⇒ sometimes called dispersion relations.

→ only properties used in proof are:
- linearity of response
- $\alpha(\omega)$ regular in UHP ⇔ CAUSALITY

→ K-K. Reln. ⇒ link, constrain structure of alpha.
⇒ constrain structure of the response.

→ ? How are K-K. Relations modified by $\alpha(\omega)$ having a pole at $\omega = 0$.

→ ? How can we exploit the structure of $\alpha_{IM}(\omega)$ to simplify the relation for $\alpha_{real}(\omega)$.

This brings us to:

3.) the Fluctuation-Dissipation Theorem

Recall definition of spectral density:

$$\begin{cases} \langle x_\omega x_{\omega'} \rangle = 2\pi (x^2)_\omega \, \delta(\omega+\omega') \\ \phi(t) = \langle x(0)\, x(t) \rangle = \int_{-\infty}^{+\infty} (x^2)_\omega e^{-i\omega t} \frac{d\omega}{2\pi} \end{cases}$$

For x interpreted as an operator $\hat{x}$:

$$2\pi (x^2)_\omega \, \delta(\omega+\omega') = \frac{1}{2} \langle \hat{x}_\omega \hat{x}_{\omega'} + \hat{x}_{\omega'} \hat{x}_\omega \rangle$$ (symmetrize order).

Now, the body which $\hat{x}$ 'acts on/measures' is assumed to be in n^{th} stationary state, so

$$\frac{1}{2}\left(\hat{X}_\omega \hat{X}_{\omega'} + \hat{X}_{\omega'} \hat{X}_\omega\right)_{nn} = \frac{1}{2}\sum_m \left[(X_\omega)_{nm}(X_{\omega'})_{mn} + (X_{\omega'})_{nm}(X_\omega)_{mn}\right]$$

diagonal matrix element ⟹ expectation

Must calculate time dependence of $\hat{X}(t)$ via matrix elements ⟹ need time-dependent wave functions.

so

$$(X_\omega)_{nm} = \int_{-\infty}^{+\infty} dt\, \underset{\text{matrix element (time independent)}}{X_{nm}}\, e^{i(\omega_{nm}+\omega)t} = 2\pi X_{nm}\,\delta(\omega_{nm}+\omega)$$

$$\omega_{nm} = (E_n - E_m)/\hbar$$

so

$$\frac{1}{2}\left(\hat{X}_\omega \hat{X}_{\omega'} + \hat{X}_{\omega'} \hat{X}_\omega\right) = 2\pi^2 \sum_m |X_{nm}|^2 \left[\delta(\omega_{nm}+\omega)\,\delta(\omega_{mn}+\omega') + \delta(\omega_{nm}+\omega')\,\delta(\omega_{mn}+\omega)\right]$$

$(X_{nm} = X_{mn}^*)$

$$= 2\pi^2 \sum_m |X_{nm}|^2 \left[\delta(\omega_{nm}+\omega)\,\delta(\omega+\omega') + \delta(\omega_{mn}+\omega)\,\delta(\omega+\omega')\right]$$

and since

$$2\pi\,(x^2)_\omega\,\delta(\omega+\omega') = \frac{1}{2}\left\langle \hat{X}_\omega \hat{X}_{\omega'} + \hat{X}_{\omega'} \hat{X}_\omega \right\rangle$$

have finally,

$$\boxed{(x^2)_\omega = \pi \sum_m |x_{nm}|^2 \left[\delta(\omega + \omega_{nm}) + \delta(\omega + \omega_{mn}) \right]}$$

→ Now, body is subject to periodic perturbation:

$$+V = -\hat{x} f = -\tfrac{1}{2}\left(f_0 e^{-i\omega t} + f_0^* e^{i\omega t} \right) \hat{x}$$

As perturbation is time-dependent, body makes transitions, and probability-per-time of the transition $n \to m$ is given by Fermi Golden Rule, i.e.

$$\underset{\substack{\uparrow \\ \text{transition} \\ \text{probability}}}{w_{m,n}} = \frac{\pi |f_0|^2}{2\hbar^2} |x_{m,n}|^2 \left\{ \delta(\omega + \omega_{m,n}) + \delta(\omega + \omega_{nm}) \right\}$$

→ Total energy absorption of body is:

$$dE/dt = Q = \sum_m w_{mn} (\hbar \omega_{mn})$$

$$Q = \frac{\pi}{2\hbar} |f_0|^2 \sum_m |X_{n,m}|^2 \left\{ \delta(\omega+\omega_{m,n}) + \delta(\omega+\omega_{n,m}) \right\} \omega_{m,n}$$

$$\boxed{Q = \frac{\pi}{2\hbar} |f_0|^2 \omega \sum_m |X_{n,m}|^2 \left\{ \delta(\omega+\omega_{n,m}) - \delta(\omega+\omega_{m,n}) \right\}}$$

using δ prop.

but $\quad Q = \frac{1}{2}\omega\, \chi_{IM}(\omega) |f_0|^2$

so

$$\boxed{\chi_{IM}(\omega) = \frac{\pi}{\hbar} \sum_m |X_{n,m}|^2 \left\{ \delta(\omega+\omega_{n,m}) - \delta(\omega+\omega_{m,n}) \right\}}$$

Note: $\chi_{IM}(\omega) \Leftrightarrow$ dissipation clearly closely related to

$$(x^2)_\omega = \pi \sum_m |X_{nm}|^2 \left\{ \delta(\omega+\omega_{nm}) + \delta(\omega+\omega_{mn}) \right\}$$

which is the fluctuation level, so

are "almost there" on F-D.T. !

Key Points: – Fermi Golden Rule

– $Q = \sum_m \hbar \omega_{mn} W_{mn}$

To complete F.-D. Thm. need relate dissipation $\alpha_{IM}(\omega)$ and fluctuation $(x^2)_\omega$ to <u>temperature</u> of body ....

to do this, average over distribution function:

i.e.

$$(x^2)_\omega = \pi \sum_{n,m} \underbrace{\rho_n}_{\exp[(\ -E_n)/T]} |x_{nm}|^2 \left\{ \delta(\omega + \omega_{nm}) + \delta(\omega + \omega_{mn}) \right\}$$

noting interchangeability of indexes, have (flip in 2nd term):

$$(x^2)_\omega = \pi \sum_{m,n} (\rho_n + \rho_m) |x_{n,m}|^2 \delta(\omega + \omega_{nm})$$

$$= \pi \sum_{m,n} \rho_n (1 + e^{-\hbar\omega_{nm}/T}) |x_{nm}|^2 \delta(\omega + \omega_{nm})$$

$$= \pi (1 + e^{-\hbar\omega/T}) \sum_{m,n} \rho_n |x_{n,m}|^2 \delta(\omega + \omega_{n,m})$$

$$\therefore \boxed{(x^2)_\omega = \pi (1 + e^{-\hbar\omega/T}) \sum_{m,n} \rho_n |x_{n,m}|^2 \delta(\omega + \omega_{n,m})}$$

and can similarly show for $\alpha_{IM}(\omega)$:

$$\boxed{\alpha_{IM}(\omega) \cong \frac{\pi}{\hbar}\left(1-e^{-\hbar\omega/T}\right)\sum_{m,n} \rho_n |X_{n,m}|^2 \delta(\omega+\omega_{n,m})}$$

and had

$$(X^2)_\omega \cong \pi\left(1+e^{-\hbar\omega/T}\right)\sum_{m,n} \rho_n |X_{n,m}|^2 \delta(\omega+\omega_{n,m})$$

$\therefore$

$$\boxed{\begin{aligned}(X^2)_\omega &= \hbar\,\alpha_{IM}(\omega)\coth\frac{\hbar\omega}{2T}\\ &= 2\hbar\,\alpha_{IM}(\omega)\left\{\tfrac{1}{2} + \tfrac{1}{e^{\hbar\omega/T}-1}\right\}\end{aligned}}$$

and, integrating over ω:

$$\boxed{\langle X^2\rangle = 2\int_0^\infty \frac{d\omega}{2\pi}(X^2)_\omega = \frac{\hbar}{\omega}\int_0^\infty \alpha_{IM}(\omega)\coth\frac{\hbar\omega}{2T}}$$

this proves F.-D. Thm!

Now, can observe:

- in general $\quad (X^2)_\omega = \hbar \, \mathrm{Im}\, \alpha(\omega) \coth \frac{\hbar\omega}{2T}$

in classical limit: $\quad \hbar\omega \ll T$

$$\coth \frac{\hbar\omega}{2T} \approx 1 \Big/ \frac{\hbar\omega}{2T} \sim \frac{2T}{\hbar\omega}$$

temperature

$$\therefore \quad \boxed{(X^2)_\omega = \frac{2T}{\omega}\, \alpha_{IM}(\omega)} \quad \rightarrow \text{classical limit of F-D. T}$$

mean square level

dissp.

and: $\quad \langle x^2 \rangle = \frac{2T}{\pi} \int_0^\infty \frac{\alpha_{IM}(\omega)}{\omega}\, d\omega$

but K-K. Reln (from α_{IM} odd) $\Rightarrow \quad \mathrm{re}\, \alpha(\omega) = \frac{2}{\pi} \int_0^\infty \frac{y\, \alpha_{IM}(y)\, dy}{(y^2 - \omega^2)}$

$$\therefore \quad \frac{2T}{\pi} \int_0^\infty \frac{\alpha_{IM}(\omega)}{\omega} = T \alpha_{real}(\omega) \Big|_0 = T \alpha_{real}(0)$$

so $\boxed{\langle x^2 \rangle = T\alpha_{real}(0)}$ → useful form

- can re-formulate theory to determine random force rather than mean square level

i.e. $(x^2)_\omega = |\alpha(\omega)|^2 (f^2)_\omega$

so $$\boxed{(f^2)_\omega = \frac{\hbar\,\alpha_{IM}(\omega)}{|\alpha(\omega)|^2} \coth\frac{\hbar\omega}{2T}}$$

→ Key Elements in Fluctuation-Dissipation Theorem

- stationarity
- causality
- T.D. P.T. → Fermi Golden Rule

and

- linearity of response.

→ An Example of an "Inappropriate" Random Process

– i.e. Consider a multiplicative process ⟺ typical of intermittency

x_j, for $j = 1, \dots N$

s/t $x_j = \begin{cases} 0 \\ 2 \end{cases}$; with $p = 1/2$, each

$$X = \prod_{j=1}^{N} x_j = \begin{cases} 2^N & \text{in one realization (all heads)} \\ 0, & \text{in all others} \end{cases}$$

and obviously 2^N realizations.

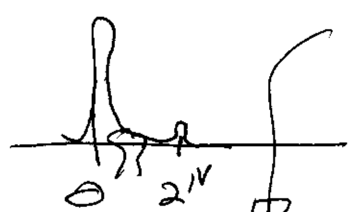
Now, compute $\langle X \rangle = \sum_{\text{real.}} X / 2N$

$= 0 + 0 + \dots + 2^N / 2^N = 1$

$\langle X^2 \rangle = \sum_{\text{real.}} X^2 / 2^N = (2^N)^2 + 0 + \dots / 2^N$

$= 2^N$

high moments weight more!

Message: → $\mathrm{pdf}(X) =$ 

→ long tail !

yet $\langle X \rangle = 1$

$\langle X^2 \rangle = 2^N$!?

→ $\langle X^p \rangle$ increases drastically with p

i.e. $\gamma_p = \log_2 \langle X^p \rangle / N = p-1$

⇔ high moments weight tail more!.

⇒ clearly a case not covered by Central Limit Theorem.

NB.: ¿ Why Are Gaussian Statistics "Ubiquitous" ?

A: → For additive processes with convergent, well behaved variances,

Central Limit Theorem ⇒ Pdf of sum, for $N \to \infty$, is Gaussian, even if Pdfs of x_i $\left(X = \sum_{i=1}^{N} x_i\right)$ are non-Gaussian!

→ Ex. Consider $X = \prod_{i=1}^{N} x_i$

where Pdf$(\ln x_i)$ well defined

i.e. $\int (\ln x_i)^2 \, \mathrm{Pdf}(\ln x_i) < \infty$

What can be said about Pdf X ?

6.) A Closer Look at Diffusion and Random Walks

- Simple Theory of Diffusion I

Consider Langevin Equation:

$$\frac{d\underline{v}}{dt} = \underbrace{-\gamma \underline{v}}_{\text{drag}} + \underbrace{\underline{\tilde{a}}(t)}_{\text{random thermal force.}}$$

further, $\langle \tilde{a}(t)\tilde{a}(t')\rangle$ "delta correlated", i.e.

$$\langle \underline{\tilde{a}}(t)\,\underline{\tilde{a}}(t')\rangle = \frac{\underline{\tilde{a}}^2}{\Delta\omega}\,\delta(t'-t)$$

(corresponds to short τ_{ac} limit)

Now,
$$\underline{v} = e^{-\gamma t}\int_0^t dt'\, e^{\gamma t'}\,\tilde{a}(t')$$

directly proceeding $\Rightarrow$

$$\langle \underline{v}(t_1)\,\underline{v}(t_2)\rangle = \int_0^{t_1}\!\!\int_0^{t_2} dt_1'\,dt_2'\, e^{-\gamma(t_1+t_2)}\, e^{\gamma(t_1'+t_2')}\,\langle \underline{\tilde{a}}(t_1')\,\tilde{a}(t_2')\rangle$$

took $t_2 > t_1$, (and need symmetrize), so:

then $\quad t_+ = (t_1 + t_2)/2 \qquad t_1 = t_+ - t_-$

$\qquad\;\; t_- = (t_2 - t_1)/2 \qquad t_2 = t_+ + t_-$

$\Big\}$ standard change-of-variables

so

$$\langle \underline{v}(t_1)\,\underline{v}(t_2)\rangle = \int_0^{t_+}\int_0^{t_-} dt_+'\,dt_-'\; 4\,e^{-2\gamma t_+}\,e^{2\gamma t_+'} \,*$$

$$\langle \underline{\tilde{q}}(t_+' - t_-')\,\underline{\tilde{q}}(t_+' + t_-')\rangle$$

$$= \int_0^{t_+}\int_0^{t_-} dt_+'\;dt_-'\;4e^{-2\gamma t_+}e^{2\gamma t_+'}\,\frac{\tilde{q}_0^{\,2}}{\Delta\omega}\,\delta(t_-')$$

$\Rightarrow$

$$\langle \underline{v}(t_1)\,\underline{v}(t_2)\rangle = 4\int_0^{t_+} dt_+'\; e^{-2\gamma t_+}\,e^{2\gamma t_+'}\,\frac{\tilde{q}_0^{\,2}}{\Delta\omega}$$

$$= \frac{2}{\gamma}\left(1 - e^{-2\gamma t_+}\right)\left(\frac{\tilde{q}_0^{\,2}}{\Delta\omega}\right)$$

$\beta t_+ \gg 1 \Rightarrow$

$$\langle \underline{\tilde{v}}(t_1)\,\underline{\tilde{v}}(t_2)\rangle = \langle \tilde{v}(0)\,\tilde{v}(\tau)\rangle \underset{\tau \gg \gamma^{-1}}{\cong} \left(\frac{2}{\gamma}\right)\frac{q_0^{\,2}}{\Delta\omega}$$

so, again have F-D. T. type result:

$$\langle v^2 \rangle = \frac{\tilde{q}_0^2}{\Delta\omega\gamma} = \tilde{q}_0^2 \tau_{ac} \tau_{damping}.$$

diffusion coefficient, in velocity

$$= \frac{D_v}{\gamma} = D_v \tau_{damp}$$

$$D_v = \tilde{q}_0^2/\Delta\omega = \tilde{q}_0^2 \tau_{ac}.$$

$$D_v = \tilde{q}_0^2/\Delta\omega = \tilde{q}_0^2 \tau_{ac}.$$

Now, for position of particle:

$$\underline{r} - \underline{r}_0 = \int_0^t \underline{v}(t')\, dt'$$

excursion, in position.

$$\underline{r} - \underline{r}_0 = \int_0^t dt' \left\{ \underline{v}_0 e^{-\gamma t'} + e^{-\gamma t'} \int_0^{t'} d\varepsilon\, e^{\gamma \varepsilon} \underset{\sim}{a}(\varepsilon) \right\}$$

and crank $\Rightarrow$

i.c.

$$\underline{r} - \underline{r}_0 = \frac{\underline{v}_0}{\gamma}(1 - e^{-\gamma t}) + \int_0^t dt' e^{-\gamma t'} \int_0^{t'} d\varepsilon\, e^{\gamma \varepsilon} \underset{\sim}{a}(\varepsilon)$$

so

$$\begin{cases} \underline{r} - \underline{r}_0 - \dfrac{\underline{v}_0}{\gamma}\left(1 - e^{-\gamma t}\right) = \displaystyle\int_0^t \psi(\varepsilon)\,\underline{\tilde{a}}(\varepsilon)\,d\varepsilon \\ \psi(\varepsilon) = \dfrac{1}{\gamma}\left(1 - e^{\gamma(\varepsilon - t)}\right) \end{cases}$$

Now, can choose/define:

- $\underline{v}_0 = 0$
- $\underline{r} - \underline{r}_0 = \delta\underline{r}$

so

$$\langle \delta\underline{r}(t_1)\, \delta\underline{r}(t_2) \rangle = \left\langle \int_0^{t_1} \psi(\varepsilon)\,\underline{a}(\varepsilon)\,d\varepsilon \int_0^{t_2} \psi(\varepsilon')\,\underline{a}(\varepsilon')\,d\varepsilon' \right\rangle$$

Taking $\langle \delta\underline{r} \cdot \delta\underline{r} \rangle$ for simplicity $\Rightarrow$

$$\langle \delta\underline{r}(0) \cdot \delta\underline{r}(\tau) \rangle \rightarrow \langle \delta r^2(\tau) \rangle$$

as before $\Rightarrow$

$$\langle \delta r(\tau)^2 \rangle = \int_0^{\tau} \psi(\varepsilon)^2\, d\varepsilon\, \langle \tilde{a}^2 \rangle\, \tau_{ac}$$

where:

$$\int_0^{\tau} \psi(\varepsilon)^2\, d\varepsilon = \frac{1}{2\gamma^3}\left(2\gamma\tau - 3 + 4e^{-\gamma\tau} - e^{-2\gamma\tau}\right)$$

so finally, have:

$$\lim_{\mathcal{T}\to\infty} \int_0^{\mathcal{T}} \Psi(\varepsilon)^2 d\varepsilon = \frac{\mathcal{T}}{\gamma^2}$$

and so

$$\lim_{\mathcal{T}\to\infty} \langle \delta r^2(\mathcal{T}) \rangle = \frac{\tilde{q}_0^2 \tilde{\tau}_{ac} \mathcal{T}}{\gamma^2}$$

and recall: $\gamma \frac{T}{m_p} = \tilde{q}_0^2 \tilde{\tau}_{ac}$ (Brownian motion in bath at Temp T)

so finally:

$$\boxed{\langle \delta r^2(\mathcal{T}) \rangle = \frac{T \mathcal{T}}{m_p \gamma} = D_x \mathcal{T}}$$

Gives Diffusion in position...

mean square excursion grows secularly.

Note:

- for particle starting from origin, mean-square particle position radius grows secularly, i.e.

$$\langle \delta r^2 \rangle \sim D_r \mathcal{T}$$

$$D_r = (T/m_p\gamma) = v_{Th}^2/\gamma = D_v/\gamma^2$$

- γ sets "mean-free-time" for random walk on $\underline{r}$

i.e. $\delta r = \int \delta v \, \tilde{\tau}$

velocity is kicked in Brownian Motion

n.b. heuristic only → can be confusing

$$\langle \delta r^2 \rangle \sim \tau_{eff}^2 \langle \delta v^2 \rangle$$
$$\sim \tau_{eff}^2 D_v \tilde{\tau} = (D_v/\gamma^2)\tilde{\tau} = D_x \tilde{\tau}.$$

- can work directly with position in long time limit, i.e.

Langevin Equation:

$$\frac{d\underline{v}}{dt} = -\gamma \underline{v} + \tilde{q}$$

with friction, $\underline{v} \to \underline{v}_{terminal}$ as $t \to \infty$, so $(t \to \infty)$

$$\underline{v} = \tilde{q}/\gamma$$

$$\Rightarrow \frac{dx}{dt} = \frac{\tilde{q}}{\gamma}$$ so as before:

relates diffusion to random force.

$$\therefore \langle \delta x^2 \rangle = D_x \tau \quad ; \quad D_x = \frac{q_0^2 \tau_{ac}}{\gamma^2}$$

Exercise: "Hamiltonian" Brownian Motion

Consider a particle in a 'randomly' fluctuating electric field, i.e.: (1D)

$$\frac{dv}{dt} = \frac{q}{m}\tilde{E}, \qquad \frac{dx}{dt} = v$$

Show $D_v = \frac{q^2}{m^2}\frac{\tilde{E}_0^{\,2}}{\Delta\omega}$, where: $\langle \delta v^2(\tau)\rangle = D_v \tau$

and $\langle \delta x(0)\delta x(\tau)\rangle = D_v \tau^3/3$

Explain physically why $\langle \delta x^2\rangle$ grows as τ^3 and contrast with dissipative Brownian Motion.

→ Simple Theory of Diffusion II

→ Diffusion and Brownian Motion have a history!

Robert Brown 1828 – discovered Brownian motion
"vitalist" interpretation

(*)
L. Bachelier, 1900 – first solution of diffusion equation

A. Einstein, 1905–1906 – physics of diffusion, Brownian motion, etc.

M. Schmoluchowski, 1906 – simplified physics

N. Wiener, 1930–1960 – mathematical foundations
(Wiener (Path) Integral)

(*) N.B.: L. Bachelier, "Theorie de la spéculation"
Ann. Sci. Ecole Norm. Sup. 17 1900
(something practical ...)

A simple way to derive the diffusion equation is to postulate:

- evolution of P (i.e. $P(\underline{x}, t)$ is probability density) is
  - → unbiased (no asymmetry)
  - → uniform/homogeneous.

for kick in x of size Δ in time τ:

Rson

$$n(t+\tau, x) = \int_{-\infty}^{+\infty} d\Delta \, n(t, x-\Delta)\, T(\tau, \Delta)$$

{density at $t+\tau$, x} {density at t, one Δ away} {transition probability i.e. probability of kick Δ in τ}

For n smooth:

$$n(t,x) + \tau \frac{\partial n}{\partial t} + \cdots = \int_{-\infty}^{+\infty} d\Delta \Big[n(t,x)\, T(\tau,\Delta) - \Delta \frac{\partial n}{\partial x}(t,x)\, T(\tau,\Delta) + \frac{1}{2}(\Delta^2) \frac{\partial^2 n}{\partial x^2}\, T(\tau,\Delta) \ldots \Big]$$

as

$$\int_{-\infty}^{\infty} d\Delta \, T(\tau,\Delta) = 1 \qquad \text{(normalization)}$$

$$\int_{-\infty}^{+\infty} d\Delta \, \Delta \, T(\tau,\Delta) = 0 \qquad \text{(no bias)}$$

and $$\int_{-\infty}^{+\infty} d\Delta\, \Delta^2\, T(\tau,\Delta) \doteq \langle\Delta^2\rangle < \infty$$

have:

$$\cancel{n(t,x)} + \tau \frac{\partial n}{\partial t} = \cancel{n(t,x)} + \frac{\langle\Delta^2\rangle}{2}\frac{\partial^2 n}{\partial x^2}$$

$$\boxed{\frac{\partial n}{\partial t} = \frac{\langle\Delta^2\rangle}{2\tau}\frac{\partial^2 n}{\partial x^2}}$$ → Diffusion Equation

$$D \doteq \frac{\langle\Delta^2\rangle}{2\tau}$$ → diffusion coefficient.

Now, diffusion equation has well known solution:

$$n(t=0, x) = \delta(x-x_0)$$ → particles concentrated at a point

then:

$$n(x,t) = \frac{1}{(2\pi D t)^{1/2}} \exp\left\{-\frac{(x-x_0)^2}{2Dt}\right\}$$

and in 2D: (r_c = cylindrical r) $\underline{x}_0 \equiv$ origin

$$n(\underline{x},t) = \left(1/2\pi t\right) \exp\left[-r_c^2/2Dt\right]$$

and in 3D:

$$n(\underline{x}, t) = 1/(2\pi t)^{3/2} \exp\left(-r^2/2Dt\right).$$

Aside: Generic Behavior of Diffusion Process varies <u>drastically</u> with dimensionality!

i.e. [In a diffusion process, rms deviation grows $\sim t^{1/2}$, but direction fluctuates (drastically). Does the particle return to the origin in 1D, 2D, 3D, and if so, how often?]

Now,

$$\begin{bmatrix}\text{Probability of} \\ \text{Return}\end{bmatrix} = \int_{t_1}^{\infty} dt \; P(0, t)$$

$\updownarrow$ PR

$P(0,t)$: {probability to find particle at origin for finite t

so

$$PR = \int_{t_1}^{\infty} dt \begin{pmatrix} 1/t^{1/2} \\ 1/t \\ 1/t^{3/2} \end{pmatrix}$$

$$\sim \begin{pmatrix} \text{Algebraically divergent} \\ \text{Logarithmically divergent} \\ \sim 1/t_1^{1/2} \end{pmatrix}$$

So 1 D → particle sure to return to start in 1 D

2 D → particle will return to start, but less likely to do so than in 1 D

3 D → $\lim_{t \to \infty} PR \to 0$

so particle wont return to origin, time asymptotically b.

→ Brownian Motion and Diffusion : Some Not-so-Simple Aspects

Path of a Brownian Particle Diffusing : Wiener Path (after N. Wiener)

$W_t(\omega)$ → random coordinate of Brownian particle

→ for given ω, W_t is deterministic

t: time

ω: realization (i.e. force pdf)

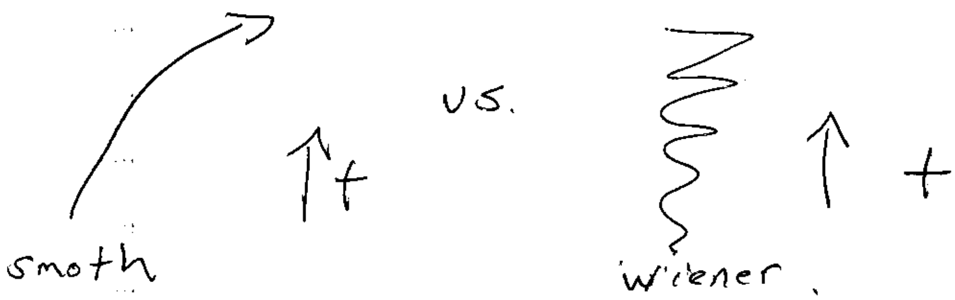
To define $W_t(\omega)$:

$$W_0(\omega) = 0$$

$W_t(\omega)$ such that: $\langle W_{t+\tau}(\omega) - W_t(\omega) \rangle = 0$

$$\langle (W_{t+\tau}(\omega) - W_t(\omega))^2 \rangle = \tau$$

∴ path is highly irregular, as for smooth path, $\mathrm{Var} \sim \tau^2$, not τ ⇒ lots of cancellation



Properties:

→ continuous

→ <u>not</u> differentiable anywhere!

<u>i.e.</u> heuristically: $\frac{dW}{dt} = \frac{\Delta W}{\Delta t}$, in sense of a limit i.e. $\frac{df}{dx} = \lim_{\Delta x \to 0} \left\{ \left(f(x+\Delta x) - f(x) \right) / \Delta x \right\}$

Now $\Delta W \sim (\Delta t)^{1/2}$ $\quad (\langle W^2 \rangle \sim DT)$

so $\frac{\Delta W}{\Delta t} \sim (\Delta t)^{-1/2} \to \infty$!

<u>i.e.</u> Wiener path is said to have derivative of $O(1/2)$.

More precisely:

Probability $|W_{t+\Delta t} - W_t| > c\Delta t$ is given by

$$P \sim 2 \int_{c\Delta t}^{\infty} \frac{dw}{(2\pi \Delta t)^{1/2}} \exp\left(-w^2/2\Delta t\right)$$

now $P \sim 2 \int_{x_0}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} \rightarrow 1$ as $x \rightarrow 0$ / $\underline{\Delta t \rightarrow 0}$!

$\underline{\text{so}}$ $P\{|W_{t+\Delta t} - W_t| > c\Delta t\} = \underline{\underline{1}}$

so derivative diverges.

Implication: Non-differentiability $\Rightarrow$ Wiener Path is "infinitely crinkly"

→ Above suggests that basic notions of calculus must be re-thought for Wiener paths

$\underline{\text{i.e.}}$ usually, $f(x(t)) \Rightarrow \frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$

$\underline{\text{so}}$ $df = f'\dot{x}\, dt$

but in diffusion $\delta x^2 \sim D\tau$!

so if $x(t)$ has $\left\{ \begin{array}{l} \text{regular} \rightarrow \text{drift} \\ \text{diffusive} \end{array} \right.$ piece

i.e.

$$x(t) = \underbrace{x_0(t)}_{\text{drift}} + \underbrace{\delta x(t)}_{\text{Wiener}}$$

then $$\frac{df}{dt} = f'\dot{x} + \frac{1}{2} f''\langle \delta \dot{x}^2 \rangle$$

$$= f'\dot{x} + \frac{D}{2} f''$$

so $$df = \left(f'\dot{x} + \frac{D}{2} f'' \right) dt$$

note factor is common.

stochastic correction
to derivative in Itô calculus.

→ Wiener Process has no memory (cf simple derivation)
i.e.

seek time correlation $\langle W_t W_s \rangle$

but $$W_t W_s = \frac{1}{2}\left[W_t^2 + W_s^2 - (W_t - W_s)^2 \right]$$

and $$\langle W_t^2 \rangle = t$$

$$\langle W_s^2 \rangle = s$$

$$\langle (W_t - W_s)^2 \rangle = |t-s|$$

(stationarity)

so

$$\langle W_t W_s \rangle = \frac{1}{2}\left[t + s - |t-s| \right]$$

$$= \min(t,s)$$

so only correlation is common, earlier time.

$$\boxed{\langle W_t W_s \rangle = \min(t,s) \longleftrightarrow \text{no memory}}$$

[Absence of memory is origin/basis/reason for non-differentiability $\leftrightarrow$ Brownian particle has no inertia ...]

Next: Kinetic Equations ....